Geometric Flows and Global Invariant Signatures

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Abstract

In this paper, we present an study on global invariant signatures of a curve based on the surface and/or perimeter evolution of the curve under the action of geometric flows. We present some results on the perimeter and surface evolution across the scales of a curve. We characterize the geometric flows invariant under similarity transformation. In the case of similarity transformations, the proposed geometric invariant is based on a scale-normalized evolution of the isoperimetric ratio of the curve. In the case of general affine geometric transformations the proposed geometric invariant is based on a scale-normalized evolution of the surface. We present also some results concerning the viscosity solutions of the involved geometric flows.

1 Introduction.

In the last years, multiscale analysis has became a common tool for many tasks in computer vision. A multiscale analysis can be defined as an operator $T_t(f)$ which provides for an original image f a sequence of images $T_t(f)$ which represent the image at a coarse scale t.

In this paper we deal with morphological multiscale analysis, which satisfy the morphological invariance, that is, the multiscale analysis $T_t(f)$ commutes with any increasing histogram modification of the image. It means that for any increasing function $g(.)$

$$T_t(f) \circ g = T_t(f \circ g)$$

the underlying hypothesis associated to this morphological invariance is that the contrast between the different objects present in the image is not important at all, and that all the information present in the image is described by the geometry of the level sets of the image. In particular, the way a shape changes under the action of a morphological multiscale analysis depends only of the geometry of its boundary.

As it was proved by Alvarez, Guichard, Lions and Morel in [1], under some minimal architectural assumptions, all the morphological and euclidean invariant multiscale analysis are generated by the partial differential equation:

$$\frac{\partial u}{\partial t} = \beta(curv(u)) \, \|\nabla u\| \tag{1}$$

where $\beta(.)$ is a nondecreasing function and $curv(u)(x,y)$ is the curvature of the level line passing by the point (x,y), that is:

$$curv(u) = div\left(\frac{\nabla u}{\|\nabla u\|}\right)$$

Therefore if $u(t,x,y)$ is the solution of equation (1) for the initial datum f, then

$$u(t,x,y) = T_t(f)(x,y)$$

Following the morphological principle, we will consider that a shape S_0 is given by a level set of the image f, that is:

$$S_0 = \overline{\{(x,y): \; f(x,y) < \lambda\}}$$

for some λ, where for a set A, we denote by $\overline{A}$ the closure of A, that is, the minimum closed set including A. We will denote by $S(t)$ the evolution across the scales of S_0, that is:

$$S(t) = \overline{\{(x,y): \; T_t(f)(x,y) < \lambda\}}$$

we will also denote by $C(t)$ the boundary of $S(t)$. In the case that $C(t)$ be a family of single Jordan curves, we can interpret the evolution of $C(t)$ in terms of curve evolution. In fact, $C(t)$ is a solution of the general geometric flow evolution equation

$$\frac{\partial C}{\partial t} = \beta(k)\vec{N} \tag{2}$$

where $\vec{N}$ represents the unit inward normal direction to the curve $C(t)$ and k is the curvature.

In the last years, there has been a lot of research devoted to the study of the evolution of plane curves using the geometric flow (2) in the particular case of $\beta(s) = s$, "*the euclidean curve shortening*" see for example [2], [3], [4], [7], [8], [9], and more recently for $\beta(s) = s^{\frac{1}{3}}$ "*the affine curve shortening*" see for example [1], [5], [12].

The main goal of this paper is to study global invariant signatures of the curves based on the surface and/or perimeter evolution of the curve under the action of a geometric flow. We point out that the function $|S(t)| = Surface(S(t))$ and $|C(t)| = Perimeter(C(t))$ are euclidean invariants of S_0. Indeed, since the multiscale analysis (1) is invariant under euclidean transformations, given 2 shapes S_0, S'_0, and $S(t)$, $S'(t)$ its corresponding evolutions, such that there exists an euclidean transformation E satisfying $S'_0 = E(S_0)$, then

$$\begin{aligned} |S(t)| = |S'(t)| \quad & for\ any\ t > 0 \\ |C(t)| = |C'(t)| \quad & for\ any\ t > 0 \end{aligned}$$

therefore the function $t \to |S(t)|$ and $t \to |C(t)|$ are euclidean invariants of the shape S_0. Equation (2) is also invariant under symmetry transformation $(x, y) \to (\pm x, \pm y)$.

The organization of the paper is as follows: In section 2, we present some results on the surface and perimeter evolution of a curve under the action of the geometric flows (2). In section 3, We characterize the geometric flows invariant under similarity transformations. In section 4, we present also some results concerning the viscosity solutions of the involved geometric flows, and the asymptotic state of the curve in some particular cases. In section 5, we analyze the global similarity invariant signature of a curve based on a scale-normalized isoperimetric ratio evolution. In section 6, we study the global affine invariant signature of a curve based on a scale-normalized surface ratio evolution.

2 Surface and perimeter evolution of a curve under the action of a geometric flow.

In this section, we will show formulas of the evolution of perimeter and surface following the equation (1). We will assume that $C(t)$, the boundary of the shape $S(t)$

at scale t, is a family of single Jordan curves. The results that we show here are a generalization of the ones presented in Gage-Hamilton [9] for the euclidean shortening flow which corresponds to the particular choice $\beta(s) = s$.

Proposition 1 *If $C(t)$ is a family of singles Jordan curves, then the evolution across the scales of the length of the curve $|C(t)|$ under the action of (1), is given by*

$$\frac{\partial |C(t)|}{\partial t} = -\int_0^{|C(t)|} k\beta(k)ds, \tag{3}$$

with s the arclength along the curve.

Remark : In the case of $\beta(k)$ be constant $(\beta(k) \equiv M)$, we have:

$$\frac{\partial |C(t)|}{\partial t} = -M\int_0^{|C(t)|} kds = -2\pi M$$

and therefore:

$$|C(t)| = |C(0)| - 2\pi Mt$$

so in particular, in this case, the evolution of the perimeter $|C(t)|$ does not depend on the geometry of $C(t)$, and then, for this particular choice of $\beta(s)$, $|C(t)|$ can not be used to discriminate between different shapes.

Proposition 2 *If $C(t)$ is a family of singles Jordan curves, then the evolution across the scales of the surface $|S(t)|$ under the action of (1), is given by*

$$\frac{\partial |S(t)|}{\partial t} = -\int_0^{|C(t)|} \beta(k)ds. \tag{4}$$

Remark : In the case of $\beta(k) = k$ we have:

$$\frac{\partial |S(t)|}{\partial t} = -\int_0^{|C(t)|} kds = -2\pi$$

and therefore:

$$|S(t)| = |S_0| - 2\pi t$$

so in particular, in this case, the evolution of the surface $|S(t)|$ does not depend on the geometry of $C(t)$, and then, $|S(t)|$ can not be used to discriminate between different shapes. We notice that this result is true only for the particular choice $\beta(s) = s$, and that in general, for other values of $\beta(s)$ the evolution of the surface depends on the geometry of S_0.

In order to proof the estimations (3) and (4) about the evolution of the perimeter and surface under the action of a morphological multiscale analysis, we will present some lemma which are a generalization of the ones presented in Gage-Hamilton [9] for the mean curvature equation which corresponds to the particular choice $\beta(s) = s$. We will assume that the evolution of the boundary of the shape, $C(t)$, is a family of simple Jordan curves, and we will denote by $C(t,u) = (x(t,u), y(t,v))$ a parametrization of the curve $C(t)$, we will also use the notation

$$v(t,u) = \left|\frac{\partial C}{\partial u}(t,u)\right| = \sqrt{\left(\frac{\partial x}{\partial u}(t,u)\right)^2 + \left(\frac{\partial y}{\partial u}(t,u)\right)^2}$$

we will assume (without loss of generality) that the interval of definition of u is $[0, 2\pi]$.

Lemma 3

$$\frac{\partial v}{\partial t} = -k\beta(k)v.$$

Proof : We compute

$$\begin{aligned}
\frac{\partial v^2}{\partial t} &= \frac{\partial}{\partial t}\langle\frac{\partial C}{\partial u}, \frac{\partial C}{\partial u}\rangle \\
&= 2\langle\frac{\partial C}{\partial u}, \frac{\partial^2 C}{\partial t \partial u}\rangle \\
&= 2\langle\frac{\partial C}{\partial u}, \frac{\partial^2 C}{\partial u \partial t}\rangle \\
&= 2\langle\frac{\partial C}{\partial u}, \frac{\partial}{\partial u}(\beta(k)\vec{N})\rangle \\
&= 2\langle\frac{\partial C}{\partial u}, \frac{\partial \beta(k)}{\partial u}\vec{N} - vk\beta(k)\vec{T}\rangle
\end{aligned}$$

using the Frenet equation $\frac{\partial \vec{N}}{\partial u} = -vk\vec{T}$. Since $\frac{\partial C}{\partial u} = v\vec{T}$, we have

$$\begin{aligned}
\frac{\partial v^2}{\partial t} &= 2\langle v\vec{T}, \frac{\partial \beta(k)}{\partial u}\vec{N} - vk\beta(k)\vec{T}\rangle \\
&= -2v^2 k\beta(k)
\end{aligned}$$

Hence we deduce $\frac{\partial v}{\partial t} = -vk\beta(k)$.

Lemma 4

$$\frac{\partial \vec{N}}{\partial t} = -\frac{\partial \beta(k)}{\partial s}\vec{T}.$$

where s is the arclength parametrization.

Proof of lemma: First, we compute the derivative of $\vec{T}$

$$\begin{aligned}
\frac{\partial \vec{T}}{\partial t} &= \frac{\partial}{\partial t}(\frac{1}{v}\frac{\partial C}{\partial u}) \\
&= \frac{\partial}{\partial t}(\frac{1}{v})\frac{\partial C}{\partial u} + \frac{1}{v}\frac{\partial^2 C}{\partial t \partial u} \\
&= -\frac{1}{v^2}\frac{\partial v}{\partial t}\frac{\partial C}{\partial u} + \frac{1}{v}\frac{\partial^2 C}{\partial u \partial t} \\
&= \frac{1}{v^2}k\beta(k)v\frac{\partial C}{\partial u} + \frac{1}{v}\frac{\partial}{\partial u}(\beta(k)\vec{N}) \\
&= k\beta(k)\vec{T} + \frac{1}{v}\frac{\partial \beta(k)}{\partial u}\vec{N} + \frac{1}{v}\beta(k)\frac{\partial \vec{N}}{\partial u} \\
&= k\beta(k)\vec{T} + \frac{1}{v}\frac{\partial \beta(k)}{\partial u}\vec{N} - \frac{1}{v}\beta(k)vk\vec{T} \text{ (by Frenet eq.)} \\
&= \frac{1}{v}\frac{\partial \beta(k)}{\partial u}\vec{N} \\
&= \frac{\partial \beta(k)}{\partial s}\vec{N}.
\end{aligned}$$

on the other hand, since $\langle \vec{N}, \vec{N} \rangle = 1$ and $\langle \vec{T}, \vec{N} \rangle = 0$ we have

$$\langle \frac{\partial \vec{N}}{\partial t}, \vec{N} \rangle = 0$$

$$\langle \frac{\partial \vec{T}}{\partial t}, \vec{N} \rangle + \langle \vec{T}, \frac{\partial \vec{N}}{\partial t} \rangle = 0$$

therefore using the previous estimations we conclude the proof of the lemma.

Next, we will present the mathematical proofs of the estimations (3) and (4).

Proof of Proposition 1: Since we have showed above $\frac{\partial v}{\partial t} = -k\beta(k)v$, so if we integrate this equality we obtain

$$\frac{\partial|C(t)|}{\partial t} = \frac{\partial}{\partial t}\left(\int_0^{2\pi} v du\right) = \int_0^{2\pi} \frac{\partial v}{\partial t} du =$$
$$= -\int_0^{2\pi} -k\beta(k)v du = -\int_0^{|C(t)|} k\beta(k) ds$$

Proof of Proposition 2: The surface $|S(t)|$ can be written as :

$$|S(t)| = -\int_0^{2\pi} \frac{1}{2}\langle C, v\vec{N}\rangle du$$

Using the previous lemma, we have:

$$\frac{\partial|S(t)|}{\partial t} = -\frac{1}{2}\int_0^{2\pi} v\beta(k) - \langle C, vk\beta(k)\vec{N}\rangle + \langle C, v\frac{\partial\beta(k)}{\partial s}\vec{T}\rangle du$$
$$= -\frac{1}{2}\int_0^{2\pi} v\beta(k) - \langle C, vk\beta(k)\vec{N}\rangle - \langle C, \frac{\partial\beta(k)}{\partial u}\vec{T}\rangle du.$$

We integrate the last term by parts:

$$\frac{\partial|S(t)|}{\partial t} = -\frac{1}{2}\int_0^{2\pi} v\beta(k) - \langle C, vk\beta(k)\vec{N}\rangle + \langle\frac{\partial C}{\partial u}, \beta(k)\vec{T}\rangle + \langle C, \beta(k)\frac{\partial\vec{T}}{\partial u}\rangle du$$
$$= -\frac{1}{2}\int_0^{2\pi} v\beta(k) - \langle C, vk\beta(k)\vec{N}\rangle + \langle v\vec{T}, \beta(k)\vec{T}\rangle + \langle C, \beta(k)vk\vec{N}\rangle du$$
$$= -\int_0^{2\pi} v\beta(k) du$$
$$= -\int_0^{L} \beta(k) ds.$$

3 Characterization of similarity invariant geometric flows.

In this section, we are going to study how to find out similarity invariant using the surface and perimeter evolution of $S(t)$. A similarity transformation H is generated by rotations, translations and isotropic scalings, and it can be expressed in the following way:

$$H(x,y) = k \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

We will say that a multiscale analysis is invariant under similarity transformations if for any transformation H, there exists a function $t \to t'(H,t)$ such that

$$H\left(T_{t'(H,t)}(f)\right) = T_t(H(f))$$

First at all, we are going to characterize the morphological multiscale analysis invariant under similarity transformation.

Proposition 5 *Let be $T_t(f)$ a morphological multiscale analysis given by (1). $T_t(f)$ is invariant under similarity transformations if and only if there exists a constant $p \geq 0$ such that:*

$$\beta(s) = \begin{cases} \beta(1)s^p & if \quad s \geq 0 \\ \beta(-1)\,(-s)^p & if \quad s < 0 \end{cases} \tag{5}$$

Moreover if k is the scaling factor of the similarity H, then

$$t'(H,t) = k^{p+1}t$$

Proof: Since the morphological multiscale analysis (1) are euclidean invariant, we can assume that the similarity transformation is given by an isotropic scaling, that is $H(x,y) = (kx,ky)$. A morphological multiscale analysis, satisfies the similarity invariant principle if and only if for any solution $u(t,x,y)$ of equation (1), the function

$$v(t,x,y) = u(t'(H,t),kx,ky)$$

is also a solution of equation (1). On the other hand

$$\frac{\partial v}{\partial t}(t,x,y) = \frac{\partial t'(H,t)}{\partial t}\frac{\partial u}{\partial t}(t'(H,t),kx,ky) = \frac{\partial t'(H,t)}{\partial t}\beta(curv(u(t,kx,ky)))\,\|\nabla u(t,kx,ky)\|$$

and

$$\beta(curv(v(t,x,y)))\,\|\nabla v(t,x,y)\| = \beta(k \cdot curv(u(t,kx,ky)))k \cdot \|\nabla u(t,kx,ky)\|$$

Then $v(t,x,y)$ is solution of (1) iff for any s

$$\frac{\partial t'(H,t)}{\partial t}\beta(s) = k\beta(ks) \tag{6}$$

First, we consider the case $s \geq 0$. We notice that if $\beta(1) = 0$ then by the above equality $\beta(k) = 0$ for any $k > 0$ and the result is trivial. In the case $\beta(1) \neq 0$, since $\frac{\partial t'(H,t)}{\partial t}$ can not depends on s, then:

$$\frac{\partial t'(H,t)}{\partial t} = k\frac{\beta(k)}{\beta(1)} \tag{7}$$

and therefore

$$t'(H,t) = k\frac{\beta(k)}{\beta(1)}t. \tag{8}$$

using equalities (6) and (7) we obtain that

$$\beta(k)\beta(s) = \beta(1)\beta(ks) \quad for\ any\ k > 0,\ s \in R$$

if we take logarithm in this equality, and we compute the derivatives with respect to s, and we evaluate in $s = 1$, we obtain:

$$\frac{\beta'(1)}{\beta(1)}\frac{1}{k} = \frac{\beta'(k)}{\beta(k)}$$

therefore, if we integrate this equality, we obtain that

$$\beta(k) = \beta(1)k^{\frac{\beta'(1)}{\beta(1)}} for\ any\ k > 0$$

then we conclude (6) taking

$$p = \frac{\beta'(1)}{\beta(1)}$$

Moreover using (8) and the previous equality we obtain:

$$t'(H,t) = k^{p+1}t$$

Now, we consider the case $s < 0$. In the same way, we obtain that if $\beta(-1) = 0$ the $\beta(-k) = 0$ for any $k > 0$ and the result is trivial, in the case $\beta(-1) \neq 0$, we obtain that

$$\beta(-k) = \beta(-1)k^{\frac{-\beta'(-1)}{\beta(-1)}} for\ any\ k > 0$$

and

$$t'(H,t) = k^{\frac{-\beta'(-1)}{\beta(-1)}+1}t$$

Therefore if $\beta(1)\beta(-1) \neq 0$ since $t'(H,t)$ does not depends on the sign of s we have that

$$p = \frac{\beta'(1)}{\beta(1)} = \frac{-\beta'(-1)}{\beta(-1)}$$

which concludes the proof.

Next we show a closed-form expression for the evolution of a circle under the action of a similarity invariant morphological multiscale analysis.

Lemma 6 *If $S_0 = B_{R_0}(x_0, y_0)$ is a circle of radius R_0,and $T_t^{(p,\beta_{-1},\beta_1)}$ is a similarity invariant morphological multiscale analysis, then, $S(t) = B_{R(t)}(x_0, y_0)$ where $R(t)$ is given by*

$$R(t) = \left(R_0^{p+1} - \beta_1(p+1)t\right)_+^{\frac{1}{p+1}}$$

Proof: Using the equivalence with the curve evolution, we have that the radius $R(t)$ of the circle satisfies the equation :

$$\frac{\partial R}{\partial t}(t) = -\beta_1 \left(\frac{1}{R(t)}\right)^p$$

and the solution of this equation is given by

$$R(t) = \begin{cases} \left(R_0^{p+1} - \beta_1(p+1)t\right)^{\frac{1}{p+1}} & if \quad t \leq \frac{R_0^{p+1}}{\beta_1(p+1)} \\ 0 & if \quad t > \frac{R_0^{p+1}}{\beta_1(p+1)} \end{cases}$$

which concludes the proof.

Remark: we notice that following the previous proposition the multiscale analysis $T_t(f)$ is not scale invariant in the sense that a similarity transformation does not modified the space and scale variables in the same way, it means that if we apply a zoom $(x,y) \to (kx,ky)$ to the shape, then the scale is modified by $t \to k^{p+1}t \neq kt$. In order to have the scale invariant property we need just to replace t by the transformation:

$$\tilde{t} = (t(p+1))^{\frac{1}{p+1}} \tag{9}$$

with the new scale $\tilde{t}$ we have that $(x, y, \tilde{t}) \to (kx, ky, k\tilde{t})$ under the action of the similarity transformation $H(x, y) = (kx, ky)$. Indeed,

$$H(T_{\tilde{t}k}(f)) = H(T_{\frac{(tk)^{p+1}}{p+1}}(f)) = T_{\frac{(t)^{p+1}}{p+1}}(H(f)) = T_{\tilde{t}}(H(f))$$

The new scale variable $\tilde{t}$ has a more physical meaning, for instance a disk of radius R_0 vanishes in a scale proportional to $\tilde{t} = R_0$. In the particular case of $\beta_1 = 1$, the vanishing scale of a circle is equal to its radius, so we can interpret that in a scale $\tilde{t}$, all the objects initially included in a disk of radius $\tilde{t}$ have been removed by the multiscale analysis.

Remark: We notice that, in fact, the similarity invariant morphological multiscale analysis depends on 3 parameters, the power $p \geq 0$ and the constant $\beta_{-1} = \beta(-1)$ and $\beta_1 = \beta(1)$ which are not completely free because the function $\beta(s)$ has to be nondecreasing. It means that if $p > 0$ then $\beta_1 \geq 0$ and $\beta_{-1} \leq 0$. In what follows we will represent the similarity invariant multiscale analysis $T_t(f)$ by these 3 parameters, that is :

$$T_t(f) = T_t^{(p,\beta_{-1},\beta_1)}(f)$$

we will use also the notation

$$S_{(p,\beta_{-1},\beta_1)}(t)$$
$$C_{(p,\beta_{-1},\beta_1)}(t)$$

to indicate the evolution of the shape S_0 and its boundary C_0 following the multiscale analysis given by (p, β_{-1}, β_1).

Among the different possibilities of similarity invariant morphological multiscale analysis let us mention 3 examples which correspond to some particular choices for (p, β_{-1}, β_1). The first example is given for the classical mathematical morphology operators dilation and erosion, which corresponds to the choices

$$(p, \beta_{-1}, \beta_1) = (0, 1, 1) \quad (10)$$
$$(p, \beta_{-1}, \beta_1) = (0, -1, -1) \quad (11)$$

The second example of multiscale analysis is given by the mean curvature motion operator, where we have typically 3 options for the choices of (p, β_{-1}, β_1):

$$(p, \beta_{-1}, \beta_1) = (1, -1, 1) \quad (12)$$
$$(p, \beta_{-1}, \beta_1) = (1, 0, 1)$$
$$(p, \beta_{-1}, \beta_1) = (1, -1, 0)$$

The third example of multiscale analysis that we consider is based on the affine invariant multiscale analysis discovered by Alvarez-Lions-Guichard-Morel [1] and Sapiro-Tannenbaum [12] in an independent way. In this case, we will use, again, 3 different choices for (p, β_{-1}, β_1):

$$
\begin{aligned}
(p, \beta_{-1}, \beta_1) &= (\frac{1}{3}, -1, 1) \\
(p, \beta_{-1}, \beta_1) &= (\frac{1}{3}, 0, 1) \\
(p, \beta_{-1}, \beta_1) &= (\frac{1}{3}, -1, 0)
\end{aligned}
\tag{13}
$$

4 Viscosity solutions

The proper framework to study the multiscale operator is the theory of viscosity solution (cf. [1]); we refer the reader to the "user's guide" of Crandall-Ishi-Lions [6] and the references inside.

Here we just want to make some remarks because the equation (1) with (5) has a possible difficulty for $Du = 0$.

For $|Du| \neq 0$, we may write:

$$
\begin{aligned}
curv(u) &= div(\frac{Du}{|Du|}) \\
&= \frac{1}{|Du|}(\Delta u - \frac{1}{|Du|^2}\sum_{i=1}^{2} \frac{\partial^2 u}{\partial x_i^2}(\frac{\partial u}{\partial x_i})^2) \\
&= \frac{1}{|Du|} Tr((Id - \frac{Du \otimes Du}{|Du|^2})D^2 u)
\end{aligned}
$$

with $q \otimes q = (q_i q_j)_{i,j}$ and Tr the trace operator.

The well-know mean curvature operator is

$$
F(q, X) = Tr((Id - \frac{q \otimes q}{|q|^2})X).
$$

Then the operator of (1) with (5) is

$$
G(q, X) = \begin{cases} \beta(1)(F(q, X))^p |q|^{1-p} & \text{if } F(q, X) \geq 0, \\ \beta(-1)(-F(q, X))^p |q|^{1-p} & \text{if } F(q, X) \leq 0. \end{cases}
$$

The extensions of F to $(0, X)$ is given by (cf. user's guide)

$$\underline{F}(q, X) = \begin{cases} F(q, X) & \text{if } q \neq 0, \\ -2||X|| & \text{if } q = 0, \end{cases} \qquad \bar{F}(q, X) = \begin{cases} F(q, X) & \text{if } q \neq 0, \\ 2||X|| & \text{if } q = 0. \end{cases}$$

Then the extension of G is

$$\underline{G}(q, X) = \begin{cases} G(q, X) & & \text{if } q \neq 0, \\ 0 & \text{if } p < 1, & \\ -2||X|| & \text{if } p = 1, & \text{if } q = 0, \\ -\infty & \text{if } p > 1, & \end{cases}$$

$$\bar{G}(q, X) = \begin{cases} G(q, X) & & \text{if } q \neq 0, \\ 0 & \text{if } p < 1, & \\ 2||X|| & \text{if } p = 1, & \text{if } q = 0. \\ +\infty & \text{if } p > 1, & \end{cases}$$

Hence, for $p \in [0, 1]$, the operator G has a upper and lower envelope bounded. In particular, for $p \in [0, 1]$, we have

$$\bar{G}(0, 0) = \underline{G}(0, 0). \tag{14}$$

The case $p > 1$ need new definition of viscosity solutions (cf. Ishii-Souganidis [11] for example).

The operator F is degenerate elliptic, i.e.,

$$F(X, q) \geq F(Y, q) \quad \text{if } X \geq Y, \tag{15}$$

for all $q \in I\!R^2$, $X, Y \in \mathcal{S}^2$ where $\mathcal{S}^2$ is the set of symmetric matrix. Then G is degenerate elliptic too.

Moreover the operator F is geometric, i.e.,

$$F(\lambda X + \mu(q \otimes q)), \lambda q) = \lambda F(X, q) \quad \text{for all } \lambda > 0 \text{ and } \mu \in I\!R. \tag{16}$$

It is straightforward that the operator G satisfies the same property.

With these three properties, we have the two following results taken from [13].

Proposition 7 *Assume (14), (15) and (16). If $u \in UC(I\!R^2)$ where UC denotes the space of uniformly continuous function, is a subsolution of (1) and v is a discontinuous supersolution of (1), and $u(., 0) \leq v(., 0)$ on $I\!R^2 \times \{0\}$ then $u(., t) \leq v(., t)$ on $I\!R^2$ for all $t > 0$.*

Proposition 8 *Assume (14), (15) and (16). Then, for any* $u_0 \in UC(\mathbb{R}^2)$*,there exists a unique solution* $u \in UC(\mathbb{R}^2)$ *of the equation.*

Using these two results, we can, for example, obtain a solution for the evolution of a disk. We approximate the discontinuous initial function u_0 by a decreasing sequence of regular functions u_0^n in $UC(\mathbb{R}^2)$. Then there exists an unique solution u^n associated to each initial data u_0^n. Using the regularity properties of viscosity solutions, the upper star limit sup of the sequence of solutions, i.e., $\limsup_{n\to+\infty, y\to x} u^n$, is a subsolution of the equation. And the lower star limit inf $(\liminf_{n\to+\infty, y\to x} u^n)$ is a supersolution. But, the functions u^n are continuous and the sequence $(u^n)_n$ is decreasing. Then $\limsup_{n\to+\infty, y\to x} u^n = \inf_n u^n$ and $\liminf_{n\to+\infty, y\to x} u^n = (\inf_n u^n)_*$ Hence $u = \inf_n u^n$ is a solution of the problem.

Concave transformation

Proposition 9 *Let* S *be a convex set. We consider the problem:*

$$\begin{cases} \dfrac{\partial u}{\partial t} = -(-(curv(u))_-)^p |Du| & \text{in } \mathbb{R}^2 \times \mathbb{R}^+, \\ u = \mathbb{1}_{\{\mathbb{R}^2 \setminus S\}} & \text{on } \mathbb{R}^2 \times \{0\}, \end{cases} \tag{17}$$

($\beta(1) = 0$ *and* $\beta(-1) = -1$ *in the definition (5)) where the function* $\mathbb{1}_{\{\mathbb{R}^2 \setminus S\}}$ *is the characteristic function of the complementary set of* S*.*

Then the stationary function $u(x,t) = \mathbb{1}_{\{\mathbb{R}^2 \setminus S\}}(x)$ *is a viscosity solution of (17).*

Proof : First we prove the property for the sub-solution.

Let $\varphi \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+)$ and (x_0, t_0) a maximum point of $u^* - \varphi$ in $\mathbb{R}^2 \times \mathbb{R}^+_*$. We may assume that $u^*(x_0, t_0) = \varphi(x_0, t_0)$ by adding $u^*(x_0, t_0) - \varphi(x_0, t_0)$ to the test-function φ.

The function u is obviously C^∞ in time variable then, by classical properties of maximum points, we have

$$\frac{\partial u}{\partial t}(x_0, t_0) = \frac{\partial \varphi}{\partial t}(x_0, t_0) = 0.$$

For the same reasons, everywhere the function u is regular at the point (x_0, t_0), the point equation (17) is satisfied by the function-test φ. The case when $D\varphi(x_0, t_0) = 0$ need the use of the theory of discontinuous Hamiltonian (cf. part on viscosity solutions).

It remains the case when x_0 is in the boundary of S, i.e, $x_0 \in \partial S$. Since the curvature of φ appears in the equation, we will investigate the level set of φ associated to the value $\varphi(x_0, t_0)$, i.e., intersection of φ with the plan $z = u^*(x_0, t_0)$, denoted by I. (In fact, we consider only the connect part of I contained (x_0, t_0), still noted I.)

If $|D\varphi(x_0, t_0)| = 0$, then I is a point or a plan around (x_0, t_0) and the equation (17) is satisfied by φ using the same arguments as before.

If $|D\varphi(x_0, t_0)| \neq 0$, then I is a curve. We have two cases following its regularity:

- If, at the point x_0, the set S have a "convex" corner, it is impossible to have a regular function such that $u^* \leq \varphi$. Remark that if the set S has a "concave" corner, it can not be a sub-solution because we are able to construct test-function with non-positive curvature at this corner.

- Then the boundary ∂S has a curvature at the point x_0, we note it $curv(S)(x_0)$. And, since $u^* \leq \varphi$, we get $curv(S)(x_0) \leq curv(\varphi)(x_0)$. Since the set S is convex, $curv(S)(x_0) \geq 0$. Hence

$$\frac{\partial \varphi}{\partial t}(x_0, t_0) = 0 \leq -(-(curv(\varphi(x_0, t_0))_-)^p |D\varphi(x_0, t_0)|$$

and it concludes the case of sub-solution.

To prove that u_* is a super-solution, let $\varphi \in C^\infty(I\!R^2 \times I\!R^+)$ and (x_0, t_0) a minimum point of $u_* - \varphi$ in $I\!R^2 \times I\!R^+_*$.

Again we obviously get

$$\frac{\partial u}{\partial t}(x_0, t_0) = \frac{\partial \varphi}{\partial t}(x_0, t_0) = 0.$$

Then the super-solution condition is

$$\frac{\partial \varphi}{\partial t}(x_0, t_0) = 0 \geq -(-(curv(\varphi(x_0, t_0))_-)^p |D\varphi(x_0, t_0)|$$

and it is empty. ◇

Corollary 10 *Let S_0 be a connect and bounded shape, and the multiscale analysis given by the equation (17). Then the asymptotic state of S_0 under the action of the above multiscale analysis is given by the convex-hull of S_0, i.e., S_0^c. That is*

$$\lim_{t \to +\infty} T_t(S_0) = S_0^c.$$

Proof : Using the preceding proposition, we have that

$$T_t(S_0^c) = S_0^c \quad \forall t > 0.$$

Since $(T_t(S_0))_{t>0}$ is increasing family sets included in S_0^c, there exists S_∞ such that

$$\lim_{t\to\infty} T_t(S_0) = \bigcup_{t>0} T_t(S_0) = S_\infty.$$

Everywhere the set S_∞ has a curvature, it is positive since $T_t(S_\infty) = S_\infty$ for any $t > 0$. Where the set S_∞ has a corner, it has to be a "convex" one by a remark did in the proof of the preceding proposition. Hence the set S_∞ is convex. But, since $S_0 \subset S_\infty$, by definition of S_0^c, we conclude that $S_0^c = S_\infty$. $\diamond$

Corollary 11 *If all the connect components of a shape S_0 are convex and S_0' is another shape which have one connect component which is non-convex, then for some $t \geq 0$ and for the multiscale analysis associated to the equation (17),*

$$|S'(t)| \neq |S(t)|.$$

Proof : Using the preceding proposition, we have that $T_t(S_0)$ is constant for all scale $t \geq 0$. But $T_t(S_0')$ will change because of the non-convex connect component; precisely it will increase. $\diamond$

Proposition 12 *Let S and S' be two regular shapes. If the evolutions of their areas are the same for all the multiscale analysis, i.e., $|S'_{(p,\beta_{-1},\beta_1)}| = |S_{(p,\beta_{-1},\beta_1)}|$ and for all scales, then they have the same perimeter.*

Proof : The assumption gives

$$-\int_0^{2\pi} k^p v du = -\int_0^{2\pi} k'^p v' du \quad \text{for all } p > 0.$$

We wish to let p go to 0 but it is an easy application of the Lebesgue's Lemma since kv is bounded by regularity of shapes. $\diamond$

5 Global Similarity Invariant Signature of a Shape.

The scale-normalized isoperimetric ratio evolution.

In order to find out similarity invariants we have to normalize the scale following the scaling factor k. First we notice that if $T_t^{(p,\beta_{-1},\beta_1)}(f)$ is a morphological multiscale analysis invariant under similarity transformations, S_0, S_0' bounded shapes and H a similarity transformation such that $H(S_0') = S_0$, then using proposition 5, and (9) we obtain

$$
\begin{aligned}
S_{(p,\beta_{-1},\beta_1)}(\tilde{t}) &= H(S'_{(p,\beta_{-1},\beta_1)}(k\tilde{t})) \quad for\ any\ t \geq 0 \\
\left|S_{(p,\beta_{-1},\beta_1)}(\tilde{t})\right| &= \frac{\left|S'_{(p,\beta_{-1},\beta_1)}(k\tilde{t})\right|}{k^2} \quad for\ any\ t \geq 0 \\
\left|C_{(p,\beta_{-1},\beta_1)}(\tilde{t})\right| &= \frac{\left|C'_{(p,\beta_{-1},\beta_1)}(k\tilde{t})\right|}{k} \quad for\ any\ t \geq 0
\end{aligned}
\tag{18}
$$

We will use as similarity invariant of a bounded shape S_0 the scale-normalized isoperimetric ratio evolution $I^{S_0}_{(p,\beta_{-1},\beta_1)}(\tilde{t})$ given by:

Definition 13 *Let be S_0 a bounded shape, we define the scale-normalized isoperimetric ratio evolution $I^{S_0}_{(p,\beta_{-1},\beta_1)}(\tilde{t})$ as the function*

$$
I^{S_0}_{(p,\beta_{-1},\beta_1)}(\tilde{t}) = 4\pi \frac{\left|S_{(p,\beta_{-1},\beta_1)}(\tilde{t}\sqrt{|S_0|})\right|}{\left|C_{(p,\beta_{-1},\beta_1)}(\tilde{t}\sqrt{|S_0|})\right|^2}
$$

We notice that $I^{S_0}_{(p,\beta_{-1},\beta_1)}(\tilde{t}) \leq 1$, and $I^{S_0}_{(p,\beta_{-1},\beta_1)}(\tilde{t}) = 1$ only in the case that $S_{(p,\beta_{-1},\beta_1)}(\tilde{t}\sqrt{|S_0|})$ be a circle. Next, we will show that $I^{S_0}_{(p,\beta_{-1},\beta_1)}(\tilde{t})$ is a similarity invariant of the shape S_0.

Theorem 14 *Let $T_t^{(p,\beta_{-1},\beta_1)}(f)$ be a morphological multiscale analysis invariant under similarity transformations, 2 bounded shapes S_0, S'_0, such that there exists a similarity transformation H with $H(S'_0) = S_0$, Then:*

$$
I^{S_0}_{(p,\beta_{-1},\beta_1)}(\tilde{t}) = I^{S'_0}_{(p,\beta_{-1},\beta_1)}(\tilde{t}) \quad for\ \tilde{t} \geq 0
$$

Proof: Let k be the scaling factor of the transformation H, using (18) we obtain:

$$
\begin{aligned}
I^{S_0}_{(p,\beta_{-1},\beta_1)}(\tilde{t}) &= 4\pi \frac{\left|S_{(p,\beta_{-1},\beta_1)}(\tilde{t}\sqrt{|S_0|})\right|}{\left|C_{(p,\beta_{-1},\beta_1)}(\tilde{t}\sqrt{|S_0|})\right|^2} = 4\pi \frac{\left|S'_{(p,\beta_{-1},\beta_1)}(\tilde{t}k\sqrt{|S_0|})\right|}{\left|C'_{(p,\beta_{-1},\beta_1)}(\tilde{t}k\sqrt{|S_0|})\right|^2} = \\
&= 4\pi \frac{\left|S'_{(p,\beta_{-1},\beta_1)}(\tilde{t}\sqrt{|S'_0|})\right|}{\left|C'_{(p,\beta_{-1},\beta_1)}(\tilde{t}\sqrt{|S'_0|})\right|^2} = I^{S_0}_{(p,\beta_{-1},\beta_1)}(\tilde{t})
\end{aligned}
$$

which concludes the proof.

Remark: We notice that in the case of the mean curvature evolution $((p, \beta_{-1}, \beta_1) = (1, -1, 1))$, if $C(t)$ is a family of single Jordan curves, then, following the results of the previous section we have that

$$\left| S_{(1,-1,1)}(\tilde{t}\sqrt{|S_0|}) \right| = |S_0| \left(1 - \pi \tilde{t}^2\right)_+$$

so we have a close form expression for the evolution of the surface. In particular we have that since $S_{(1,-1,1)}(t)$ converges towards a circle before vanishing then $I^{S_0}_{(1,-1,1)}(\tilde{t})$ satisfies:

$$\underset{\tilde{t} \to \left(\sqrt{\frac{1}{\pi}}\right)^-}{Lim} I^{S_0}_{(1,-1,1)}(\tilde{t}) = 1.$$

On the other hand, in the case $p = 0$ we have that

$$\left| C_{(p,\beta_{-1},\beta_1)}(\tilde{t}\sqrt{|S_0|}) \right| = \left(|C_0| - 2\pi\beta_1 \tilde{t}\sqrt{|S_0|}\right)_+$$

so we have a close form expression for the evolution of the perimeter.

Remark: In practice, in the case of nonconvex curves, the possibility of taking different values for β_1 and β_{-1} is very important because we can discriminate between the convex and concave part of the curve, in fact, we conjecture that the isoperimetric evolution characterize the curves in the following way:

CONJECTURE: given 2 Jordan curves C and C', if the scale-normalized isoperimetric ratio is the same for both curves for any scale t and for any similarity invariant geometric flow, then there exist a similarity transformation H such that $H(C) = C'$ (modulus a symmetry)

6 Global Affine Invariant Signature of a Shape.

We consider a general affine transformation given by

$$H(x, y) = A \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

where A is a $2x2$ matrix with $|A| \neq 0$

In [1], it was showed that the only affine invariant morphological multiscale analysis is given by

$$\beta(s) = \begin{cases} \beta_1 s^{\frac{1}{3}} & if \quad s \geq 0 \\ \beta_{-1}(-s)^{\frac{1}{3}} & if \quad s < 0 \end{cases}$$

where $\beta_1 \geq 0$ and $\beta_{-1} \leq 0$. In this case we have that

$$H\left(T_{t'(H,t)}(f)\right) = T_t(H(f))$$

where

$$t'(H,t) = |A|^{\frac{4}{3}}\, t$$

On the other hand, given 2 bounded shapes S_0, S'_0, such that there exists an affine transformation H with $H(S'_0) = S_0$, we have that:

$$\left|S_{(p,\beta_{-1},\beta_1)}(\tilde{t})\right| = \frac{\left|S'_{(p,\beta_{-1},\beta_1)}(\sqrt{|A|}\tilde{t})\right|}{|A|} \quad for\ any\ t \geq 0 \tag{19}$$

In the case of the affine invariant representation, we can not use the scale-normalized isoperimetric ratio because the perimeter is not invariant under affine transformations. So we will propose a geometric invariant based just on the surface evolution. We will introduce the scale-normalized surface ratio.

Definition 15 *Let be S_0 a bounded shape, we define the scale-normalized surface ratio evolution $SR^{S_0}_{(p,\beta_{-1},\beta_1)}(\tilde{t})$ as the function*

$$SR^{S_0}_{(p,\beta_{-1},\beta_1)}(\tilde{t}) = \frac{\left|S_{(p,\beta_{-1},\beta_1)}(\tilde{t}\sqrt{|S_0|})\right|}{|S_0|}$$

Next, we will show that $SR^{S_0}_{(p,\beta_{-1},\beta_1)}(\tilde{t})$ is an affine invariant of the shape S_0.

Theorem 16 *Let $T^{(1,\beta_{-1},\beta_1)}_t(f)$ be a morphological multiscale analysis invariant under affine transformations ($p = \frac{1}{3}$), 2 bounded shapes S_0, S'_0, such that there exists an affine transformation H with $H(S'_0) = S_0$, Then:*

$$SR^{S_0}_{(\frac{1}{3},\beta_{-1},\beta_1)}(\tilde{t}) = SR^{S'_0}_{(\frac{1}{3},\beta_{-1},\beta_1)}(\tilde{t}) \quad for\ \tilde{t} \geq 0$$

Proof: Using (19) we obtain:

$$SR^{S_0}_{(\frac{1}{3},\beta_{-1},\beta_1)}(\tilde{t}) = \frac{\left|S_{(\frac{1}{3},\beta_{-1},\beta_1)}(\tilde{t}\sqrt{|S_0|})\right|}{|S_0|} = \frac{\left|S'_{(\frac{1}{3},\beta_{-1},\beta_1)}(\tilde{t}\sqrt{|S_0|\,|A|})\right|}{|S_0|\,|A|} =$$

$$= \frac{\left|S'_{(\frac{1}{3},\beta_{-1},\beta_1)}(\tilde{t}\sqrt{|S'_0|})\right|}{|S'_0|} = SR^{S'_0}_{(\frac{1}{3},\beta_{-1},\beta_1)}(\tilde{t})$$

which concludes the proof.

Remark: The following question remain unanswered:

Given 2 Jordan curves C and C', if the scale-normalized surface ratio is the same for both curves for any scale t and for any affine invariant geometric flow, then there exist an affine transformation H such that $H(C) = C'$?.

The vanishing scale.

Definition 17 *Given a morphological multiscale analysis $T_t^{(p,\beta_{-1},\beta_1)}(f)$ with $p \geq 0$ and a shape S_0, we define the vanishing scale $t_\infty^{(p,\beta_{-1},\beta_1)}(S_0)$ as the number:*

$$t_\infty^{(p,\beta_{-1},\beta_1)}(S_0) = \sup_{t>0}\{|S(t)| > 0\}$$

Remark: We notice that if $\beta_1 > 0$ and S_0 is a bounded shape, then by the inclusion principle we have that

$$S(t) \subset B_{\left(R_0^{p+1}-\beta_1 t(p+1)\right)_+^{\frac{1}{p+1}}}(x_0,y_0)$$

and therefore $t_\infty^{(p,\beta_{-1},\beta_1)}(S_0) < R_0^{p+1}/(\beta_1(p+1)) < \infty$.

Remark: We notice that if we use the scale $\tilde{t}$, given in (9), instead of t, then the vanishing scale for a circle of radius R_0 is given by:

$$\tilde{t}_\infty^{(p,\beta_{-1},\beta_1)}(B_{R_0}(x_0,y_0)) = \beta_1^{\frac{1}{p+1}} R_0$$

On the other hand, under the action of a similarity or affine transformation $H()$ we have that $\tilde{t}_\infty^{(p,\beta_{-1},\beta_1)} \rightarrow k\tilde{t}_\infty^{(p,\beta_{-1},\beta_1)}$,where k is the scaling factor of the similarity or $\tilde{t}_\infty^{(p,\beta_{-1},\beta_1)} \rightarrow \sqrt{|A|}\tilde{t}_\infty^{(p,\beta_{-1},\beta_1)}$ in the case of an affine transformation. This relations provides to us another way to normalize the scale in the scale-normalized isoperimetric ratio and the scale-normalized surface ratio, in other words the following functions are similarity (respect. affine) invariant of a shape

$$I^{S_0}_{(p,\beta_{-1},\beta_1)}(\tilde{t}) = \frac{\left|S_{(p,\beta_{-1},\beta_1)}(\tilde{t}(\tilde{t}_\infty^{(p,\beta^*_{-1},\beta^*_1)}))\right|}{\left|C_{(p,\beta_{-1},\beta_1)}(\tilde{t}(\tilde{t}_\infty^{(p,\beta^*_{-1},\beta^*_1)}))\right|^2}$$

$$SR^{S_0}_{(\frac{1}{3},\beta_{-1},\beta_1)}(\tilde{t}) = \frac{\left|S_{(\frac{1}{3},\beta_{-1},\beta_1)}(\tilde{t}(\tilde{t}_\infty^{(\frac{1}{3},\beta^*_{-1},\beta^*_1)}))\right|}{\left(\tilde{t}_\infty^{(\frac{1}{3},\beta^*_{-1},\beta^*_1)}\right)^2}$$

so we can compute $\tilde{t}_\infty^{(p,\beta^*_{-1},\beta^*_1)}$ for a particular choice of $(\beta^*_1, \beta^*_{-1})$ and then we can normalize the scale with $\tilde{t}_\infty^{(p,\beta^*_{-1},\beta^*_1)}$ for any other multiscale analysis (β_1, β_{-1})

Erosion

Proposition 18 *Using the multiscale analysis given by the equation:*

$$u_t = -|Du|, \tag{20}$$

let S be a bounded shape and t_∞ its vanishing scale.

If the vanishing scale is equal to

$$t_\infty = \sqrt{\frac{|S|}{\pi}},$$

then the set S is a ball of radius t_∞.

Proof : The keystone of the demonstration is the following fact:

$$\text{if } x \in T_t(S), \text{ then } \mathcal{B}(x,t) \subset S,$$

where $\mathcal{B}(x,t)$ is a ball of center x and radius t. This property is an immediate consequence of the "erosion" equation (20).

We note S_∞ the limit set of the family of non-empty set $(T_t(S))_{t_\infty > t > 0}$ given by

$$S_\infty = \bigcap_{t_\infty > t > 0} \overline{T_t(S)}.$$

This set is not empty since $\mathbb{R}^2$ is a Cauchy space.

Let the set $\tilde{S}$ defined by

$$\tilde{S} = \bigcup_{y \in S_\infty} \mathcal{B}(y, t_\infty).$$

By the property remembered in the beginning, the set $\tilde{S}$ is included in S. But, for the compatibility of their surfaces, we conclude that the set S_∞ is a singleton, i.e, $\{y\}$ and $S = \mathcal{B}(y, t_\infty)$. $\diamond$

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References

[1] L. Alvarez, F. Guichard, P.-L. Lions and J.-M. Morel, *Axioms and fundamental equations in image processing*, Arch. Rational Mech. Anal., Vol. 123, 199–257, 1993.

[2] L. Alvarez and J.M. Morel, *Formalization and Computational Aspects of Image Analysis,* Acta Numerica pp. 7-59, 1994.

[3] S.Angenent, *Parabolic equations for curves on surfaces, Part I. Curves with p-integrable curvature*, Annals of Mathematics Vol. 132, 451-483, 1990.

[4] S.Angenent, *Parabolic equations for curves on surfaces, Part II. Intersecciions blow-up, and generalized solutions*, Annals of Mathematics Vol. 133, 171-215, 1991.

[5] S.Angenent, G.Sapiro and A.Tannenbaum, *On the affine heat equation for non-convex curves*, Journal of the American Mathematical Society Vol. 11, no. 3, 601–634, 1998.

[6] M.G. Crandall, H.Ishii, P.-L. Lions User's guide to viscosity solutions of second order partial differential equations, *Bull. AMS* 27, 1-67.

[7] M.Gage, *An isoperimetric inequality with applications to curve shortening*, Duke Mathematical Journal , Vol. 50, No. 4, 1225-1229, 1983.

[8] M.Gage, *Curve shortening makes convex curves circular*, Inventiones Mathematicae, Vol. 76, 357-364, 1984.

[9] M. Gage and R.S. Hamilton, *The heat equation shrinking convex plane curves*, J. Differential Geometry, Vol. 23, 69–96, 1986.

[10] M. Grayson, *The heat equation shrinks embedded plane curves to round points*, J. Differential Geometry, Vol 26, 285–314, 1987.

[11] Ishii H. & Souganidis P. *Generalized motion of noncompact hypersurfaces with velocity having arbitrary growth on the curvature tensor* Tohoku mathematical journal, 47(2), June 1995, 227-250

[12] G.Sapiro and A.Tannenbaum, *Affine invariant scale space*, International Journal of Computer Vision, Vol 11, 25–44, 1993.

[13] Souganidis P. *Front propagation: Theory and Applications*, CIME Course on Viscosity Solutions, Lect. Notes in Math. 1660, Springer-Verlag, 1997.