A PDE model for computing the Optical Flow

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Abstract

In this paper we present a new model for optical flow calculation using a variational formulation which preserves discontinuities of the flow much better than classical methods. We study the Euler-Lagrange equations asociated to the variational problem. In the case of quadratic energy, we show the existence and uniqueness of the corresponding evolution problem. Since our method avoid linearization in the optical flow constraint, it can recover large displacement in the scene. We avoid convergence to irrelevant local minima by embedding our method into a linear scale-space framework and using a focusing strategy from coarse to fine scales.

Introduction

Optical Flow computation is a key problem in artificial vision. It consists of finding the motion of objects in a sequence of images. We shall consider 2 images $I_1(x, y)$ and $I_2(x, y)$ (defined on IR^2 to simplify the discussion) which represent 2 consecutive views in a sequence of images. Determining the optical flow is then finding a function $h(x, y) = (u(x, y), v(x, y))$ such that :

$$I_1(x, y) \approx I_2(x + u(x, y), y + v(x, y)),\ \forall (x, y) \in IR^2$$

In general this problem has an infinite number of solutions. Take for example a sequence representing a black disk moving on a white background. In this case any function $(x, y) + h(x, y)$ associating a point of the black disk in the first image to a point on the black disk in the second image, and a point of the background in the first image to another point on the background in the second image satisfies the last equality for all points (x, y). Notice that we only best can compute the apparent motion, i.e. the motion in the direction normal to the disk boundary. The possible circular motion (a rotation leaves the disk unchanged) is totally undetectable.

To compute $h(x, y)$ the preceeding equality is usually linearized yielding the so-called "optical flow constraint"

$$I_1(\overline{x}) - I_2(\overline{x}) \approx \langle \nabla I_2(\overline{x}), h(\overline{x}) \rangle \qquad \forall (x, y) \in IR^2,$$

where $\overline{x} = (x, y)$. If at each point we suppose that the motion is only in the perpendicular direction to the level line passing through this point, i.e. $h(\overline{x}) = k(\overline{x})\nabla I_1(\overline{x})$, then we deduce from last equation that when $\langle \nabla I_2(\overline{x}), \nabla I_1(\overline{x}) \rangle \neq 0$, then :

$$k(\overline{x}) = \frac{I_1(\overline{x}) - I_2(\overline{x})}{\langle \nabla I_2(\overline{x}), \nabla I_1(\overline{x}) \rangle}.$$

Unfortunately this last equality is too local, and allows only for estimating motions of the order of one pixel. Indeed in images sequences, objects move with "a priori" unpredictable velocities, and thus there can be an important displacement of the objects between two consecutive images. In order to estimate these large displacements it is necessary to introduce a scale factor for the fusion of information before computing the flow $h(\overline{x})$. A common way of doing this is to convolve both images with a gaussian kernel $G_\sigma(\overline{x})$ (where σ is the standard deviation) before computing the motion. In other words we study equation :

$$G_\sigma * (I_1 - I_2)(\overline{x}) \approx \langle \nabla (G_\sigma * I_2)(\overline{x}), h(\overline{x}) \rangle, \qquad \forall (x,y) \in R^2.$$

Notice that this model does not impose any regularity condition on the solution $h(\overline{x})$. In this work we propose a new model for computing $h(\overline{x})$ via the minimum of the following energy functional :

$$\begin{aligned} E(h) =& \frac{1}{2} \int_{IR^2} \left(I_1(\overline{x}) - I_2(\overline{x} + h(\overline{x}))\right)^2 dx \\ &+ C \int_{IR^2} g\left(\|\nabla I_1\|\right) \Phi\left(\|\nabla u(\overline{x})\|\right) dx \\ &+ C \int_{IR^2} g\left(\|\nabla I_1\|\right) \Phi\left(\|\nabla v(\overline{x})\|\right) dx, \end{aligned}$$

where $h(\overline{x}) = (u(\overline{x}), v(\overline{x}))$, C is a positive constant, g is a strictly positive, decreasing function, and Φ is an increasing function with $\Phi(0) = 0$. This formulation preserves discontinuities of the flow much better than the classical one introduced in [4] where $g \equiv 1$ and $\Phi(s) = s^2$. The associated Euler-Lagrange equations of the above functional give rise to the following system of partial differential equations :

$$\begin{array}{l} \left(I_1(\overline{x}) - I_2(\overline{x} + h(\overline{x}))\right) \frac{\partial I_2}{\partial x}(\overline{x} + h(\overline{x}) + C div \left(g\left(\|\nabla I_1\|\right) \frac{\Phi'(\|\nabla u\|)}{\|\nabla u\|} \nabla u \right) = 0 \\ \left(I_1(\overline{x}) - I_2(\overline{x} + h(\overline{x}))\right) \frac{\partial I_2}{\partial y}(\overline{x} + h(\overline{x}) + C div \left(g\left(\|\nabla I_1\|\right) \frac{\Phi'(\|\nabla v\|)}{\|\nabla v\|} \nabla v \right) = 0 \end{array}$$

In this work we shall study the solutions to this system of partial differential equations for the determination of the optical flow. More specifically we will consider as good candidates the asymptotic solutions of the associated evolution problem for different choices of the function Φ. Some natural candidates for the fuction Φ are:

$$\begin{aligned} \Phi(s) &= \frac{s^2}{2} \\ \Phi(s) &= s \end{aligned}$$

the first choice provides the quadratic classical functional, and the second choice provides the total variation functional. In this paper we will focus our attention in the quadratic case.

The quadratic case

Let us consider in this section the case when $\Phi(x) = x^2/2$. In this case we will study the following parabolic problem :

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} = Cdiv\left(g\left(\|\nabla I_1\|\right)\nabla u\right) + \left(I_1 - I_2(Id+h)\right)\frac{\partial I_2}{\partial x}(Id+h) \\ \frac{\partial v}{\partial t} = Cdiv\left(g\left(\|\nabla I_1\|\right)\nabla v\right) + \left(I_1 - I_2(Id+h)\right)\frac{\partial I_2}{\partial y}(Id+h) \end{array} \right. \quad (1)$$

with initial condition $(u(0), v(0)) = h_0$ and seeing functions I_1 and I_2 as two periodized consecutive images (which are defined on a unit square for example) in a sequence of images.

Abstract framework

Let $H = L^2(IR^2) \times L^2(IR^2)$, and let us denote by $A : D(A) \subset H \to H$ the maximal monotone operator (C and g are strictly positive) defined by :

$$A(h) = -C \left(\begin{array}{c} div\left(g\left(\|\nabla I_1\|\right)\nabla u\right) \\ div\left(g\left(\|\nabla I_1\|\right)\nabla v\right) \end{array} \right),$$

and by $F : H \to H$ the function defined by :

$$F(h) = \left(I_1 - I_2(Id+h)\right)\nabla I_2(Id+h).$$

Then the abstract evolution problem writes :

$$\left\{ \begin{array}{l} \frac{dh}{dt} + Ah = F(h) \text{ in } H, \forall t \in [0,T] \\ h(0) = h_0 \text{ in } H \end{array} \right. \quad (2)$$

where H is a Hilbert space, $F : H \to H$ is continuous and $A : D(A) \subset H \to H$ is a maximal monotone operator. Any classical solution $h \in C^1([0,T]; H) \cap C([0,T]; D(A))$ of (2), where the norm on $D(A)$ is defined by $\|h\|_{D(A)} = \|h\|_H + \|Ah\|_H$, writes :

$$h(t) = S(t)h_0 + \int_0^t S(t-s)F(h(s))ds, \quad (3)$$

where $\{S(t)\}_{t>0}$ is the contractions semi-group associated to the homogeneous problem.

Definition 1 *We shall say that $h \in C([0,T]; H)$ is a generalized solution of (2) if it satisfies (3).*

Existence and uniqueness result

We shall start with the following preliminary result.

Lemme 2 *Suppose that $I_2 \in W^{1,\infty}(IR^2)$ and $I_1 \in L^\infty(IR^2)$. Then, F is L-Lipschitz, for some constant L depending on functions I_1 and I_2.*

Proof 3 *Let $h_1, h_2 \in H$. For the i-th component of $F(h_1) - F(h_2)$, $i = 1, 2$, we have*

$$\begin{aligned}
|F_i(h_1) - F_i(h_2)| = &|(I_1 - I_2(Id + h_1))\partial_i I_2(Id + h_1) \\
&- (I_1 - I_2(Id + h_2))\partial_i I_2(Id + h_2)|, \\
&\leq |I_2(Id + h_1)\partial_i I_2(Id + h_1) - I_2(Id + h_2)\partial_i I_2(Id + h_2)| \\
&+ |I_1|.|\partial_i I_2(Id + h_1) - \partial_i I_2(Id + h_2)|, \\
&\leq \frac{1}{2}|\partial_i(|I_2|^2)(Id + h_1) - \partial_i(|I_2|^2)(Id + h_2)| \\
&+ \|I_1\|_\infty.|\partial_i I_2(Id + h_1) - \partial_i I_2(Id + h_2)|, \\
&\leq \frac{1}{2}C_{Lip}(\partial_i(|I_2|^2)).|h_1 - h_2| + \|I_1\|_\infty.C_{Lip}(\partial_i I_2).|h_1 - h_2| \\
&\leq \left(\frac{1}{2}C_{Lip}(\partial_i(|I_2|^2)) + \|I_1\|_\infty.C_{Lip}(\partial_i I_2)\right).|h_1 - h_2|,
\end{aligned}$$

where $C_{Lip}(f)$ denotes the Lipschitz constant of function f. We finally deduce that :

$$\begin{aligned}
|F(h_1) - F(h_2)|_H &= |F_1(h_1) - F_1(h_2)|_H + |F_2(h_1) - F_2(h_2)|_H \\
&\leq \sum_{i=1}^{2}\left(\frac{1}{2}C_{Lip}(\partial_i(|I_2|^2)) + \|I_1\|_\infty.C_{Lip}(\partial_i I_2)\right).|h_1 - h_2|_H.
\end{aligned}$$

we can conclude letting

$$L = \sum_{i=1}^{2}\left(\frac{1}{2}L\left(\partial_i(|I_2|^2)\right) + \|I_1\|_\infty.C_{Lip}(\partial_i I_2)\right).$$

We now can state the existence and unicity result for problem (1).

Theorem 4 *If $I_2 \in W^{1,\infty}(IR^2)$ and $I_1 \in L^\infty(IR^2)$, then, for all $h_0 \in H$ there exists a unique generalized solution $h(t) \in C([0, \infty[; H)$ of (1). Moreover, if $h_0 \in D(A)$, then the solution is a classic one.*

Proof 5 *In view of hypothesis on I_1 and I_2 we can apply Lemma 2. Assume $h_1(t)$ and $h_2(t)$ are solutions of (3) for initial conditions $h_1(0)$ and $h_2(0)$ we then have, using the fact that $-A$ is dissipative (which yields $|S(t)f|_H \leq |f|_H$), and the Lipschitz continuity of F :*

$$|h_1(t) - h_2(t)|_H \leq |h_1(0) - h_2(0)|_H + L\int_0^t |h_1(s) - h_2(s)|_H ds.$$

Now applying Gronwall Lemma, we have :

$$|h_1(t) - h_2(t)|_H \leq e^{Lt}.|h_1(0) - h_2(0)|_H,$$

which yields unicity of the solution if it exists. Now consider the Banach space defined by

$$E = \{h \in C([0, \infty[; H) \ : \ \sup_{t\geq 0}|h(t)|_H e^{-Kt} < \infty\}$$

endowed with the norm $\|h\| = \sup_{t\geq 0} |h(t)|_H e^{-Kt}$. *Let* $\phi : E \to C([0,\infty[; H)$ *defined by :*

$$\phi(h)(t) = S(t).h_0 + \int_0^t S(t-s)F(h(s))ds.$$

If $K > L$, *then* $\phi(E) \subset E$ *and* ϕ *is* $\frac{L}{K}$*-Lipschitz since :*

$$\begin{aligned}
\|\phi(h_1) - \phi(h_2)\| &= \sup_{t\geq 0} |\phi(h_1)(t) - \phi(h_2)(t)|_H e^{-Kt}, \\
&\leq \sup_{t\geq 0} \int_0^t L|h_1(s) - h_2(s)|_H ds e^{-Kt}, \\
&\leq \sup_{t\geq 0} L\|h_1 - h_2\|.e^{-Kt} \int_0^t e^{Ks} ds, \\
&\leq \sup_{t\geq 0} \frac{L}{K}\|h_1 - h_2\|.e^{-Kt}(e^{Kt} - 1), \\
&\leq \frac{L}{K}\|h_1 - h_2\|.
\end{aligned}$$

We deduce that ϕ *is a contraction and by Banach fixed point theorem there exists a unique* h *such that* $\phi(h) = h$, *which is a generalized solution of (1). Now let* $h_0 \in D(A)$, $\tau > 0$ *and* $t > 0$. *Let us recall that*

$$|h(t+\tau) - h(t)|_H \leq e^{Lt}|h(\tau) - h_0|_H. \tag{4}$$

On the other hand we have :

$$\begin{aligned}
|h(\tau) - h(0)|_H &\leq |S(\tau)h_0 - h_0|_H + \int_0^\tau |F(h_0)|_H ds + L\int_0^\tau |h(s) - h(0)|_H ds, \\
&\leq \tau(\|A\|.|h_0|_H + |F(h_0)|_H) + L\int_0^\tau |h(s) - h(0)|_H ds,
\end{aligned}$$

and we deduce from Gronwall Lemma and the fact that $e^{Lt} - 1 \leq Lte^{Lt}$, $\forall t \geq 0$, *that :*

$$\begin{aligned}
|h(\tau) - h(0)|_H &\leq \frac{e^{L\tau} - 1}{L}(\|A\|.|h_0|_H + |F(h_0)|_H), \\
&\leq \tau e^{L\tau}(\|A\|.|h_0|_H + |F(h_0)|_H).
\end{aligned} \tag{5}$$

>*From (4) and (5) we get, for all* $t, t' \in [0, T]$,

$$|h(t) - h(t')|_H \leq e^{2LT}(\|A\|.|h_0|_H + |F(h_0)|_H)|t - t'|.$$

$h(t)$ *being lipschitz, so is* $t \to F(h(t))$, *and following some classical results (see for instance [2]) we can conclude that the solution is a classical one.*

A Linear Scale-Space Approach to Recover Large Displacements

In general, the Euler-Lagrange equations will have multiple solutions. As a consequence, the asymptotic state of the parabolic system , which we use for approximating the optical flow, will depend on the initial data. Typically, the convergence is the better, the closer the initial data is to the asymptotic state. When we expect small displacements in the scene, the natural choice is to take $u \equiv v \equiv 0$ as initialization of the flow. For large displacement fields, however, this will not work, and we need better initial data. To this end, we embed our method into a linear scale-space framework. Considering the problem at a coarse scale avoids that the algorithm gets trapped in physically irrelevant local minima. The coarse-scale solution serves then as initial data for solving the problem at a finer scale. Scale focusing has a long tradition in linear scale-space theory (see e.g. Bergholm [1] for an early approach. Detailed descriptions of linear scale-space theory can be found in [3], [5].

We proceed as follows. First, we introduce a linear scale factor in the parabolic PDE system in order to end up with

$$\begin{aligned}\frac{\partial u_\sigma}{\partial t} &= C div\left(g\left(\left|\,\|\nabla G_\sigma * I_1\|\,\right|\right)\frac{\Phi'(\|\nabla u\|)}{\|\nabla u\|}\nabla u\right) + \\ &+ \Big(G_\sigma * I_1(\overline{x}) - G_\sigma * I_2(\overline{x} + h_\sigma(\overline{x}))\Big)\frac{\partial\left(G_\sigma * I_2\right)}{\partial x}(\overline{x} + h_\sigma(\overline{x})), \qquad (6)\\ \frac{\partial v_\sigma}{\partial t} &= C div\left(g\left(\|\nabla G_\sigma * I_1\|\right)\frac{\Phi'(\|\nabla v\|)}{\|\nabla v\|}\nabla v\right) + \\ &+ \Big(G_\sigma * I_1(\overline{x}) - G_\sigma * I_2(\overline{x} + h_\sigma(\overline{x}))\Big)\frac{\partial\left(G_\sigma * I_2\right)}{\partial y}(\overline{x} + h_\sigma(\overline{x})), \qquad (7)\end{aligned}$$

where $G_\sigma * I$ represents the convolution of I with a Gaussian of standard deviation σ.

The convolution with a Gaussian blends the information in the images and allows us to recover a connection between the objects in I_1 and I_2. We start with a large initial scale σ_0. Then we compute the optical flow $(u_{\sigma_0}, v_{\sigma_0})$ at scale σ_0 as the asymptotic state of the solution of the above PDE system using as initial data $u \equiv v \equiv 0$. Next, we choose a number of scales $\sigma_n < \sigma_{n-1} < .... < \sigma_0$, and for each scale σ_i we compute the optical flow $(u_{\sigma_i}, v_{\sigma_i})$ as the asymptotic state of the above PDE system with initial data $(u_{\sigma_{i-1}}, v_{\sigma_{i-1}})$. The final computed flow corresponds to the smallest scale σ_n. In accordance with the logarithmic sampling strategy in linear scale-space theory, we choose $\sigma_i := \eta^i \sigma_0$ with some decay rate $\eta \in (0,1)$.

Numerical Scheme

The numerical scheme follows the general framework described in [6]. We discretize the parabolic system (6)–(7) by finite differences. Derivatives in x and y are approximated by central differences, and for the discretization in t direction we use an explicit (Euler forward) scheme. Gaussian convolution was performed in the spatial domain with renormalized Gaussians, which where truncated at 5 times their standard deviation.

Then our explicit scheme has the structure

$$\begin{aligned}
\frac{u_{i,j}^{k+1}-u_{i,j}^{k}}{\tau} &= C\left(\frac{g_{i+1,j}^{k}+g_{i,j}^{k}}{2}\,\frac{u_{i+1,j}^{k}-u_{i,j}^{k}}{h_1^2}+\frac{g_{i-1,j}^{k}+g_{i,j}^{k}}{2}\,\frac{u_{i-1,j}^{k}-u_{i,j}^{k}}{h_1^2}+\right.\\
&\quad\left.+\frac{g_{i,j+1}^{k}+g_{i,j}^{k}}{2}\,\frac{u_{i,j+1}^{k}-u_{i,j}^{k}}{h_2^2}+\frac{g_{i,j-1}^{k}+g_{i,j}^{k}}{2}\,\frac{u_{i,j-1}^{k}-u_{i,j}^{k}}{h_2^2}\right)+\\
&\quad+\Big(I_{1,\sigma}(\overline{x}_{i,j}-\overline{h}_{\sigma,i,j}^{k})-I_{2,\sigma}(\overline{x}_{i,j})\Big)\;I_{1,x,\sigma}(\overline{x}_{i,j}-\overline{h}_{\sigma,i,j}^{k}), \qquad (8)\\
\frac{v_{i,j}^{k+1}-v_{i,j}^{k}}{\tau} &= C\left(\frac{g_{i+1,j}^{k}+g_{i,j}^{k}}{2}\,\frac{v_{i+1,j}^{k}-v_{i,j}^{k}}{h_1^2}+\frac{g_{i-1,j}^{k}+g_{i,j}^{k}}{2}\,\frac{v_{i-1,j}^{k}-v_{i,j}^{k}}{h_1^2}+\right.\\
&\quad\left.+\frac{g_{i,j+1}^{k}+g_{i,j}^{k}}{2}\,\frac{v_{i,j+1}^{k}-v_{i,j}^{k}}{h_2^2}+\frac{g_{i,j-1}^{k}+g_{i,j}^{k}}{2}\,\frac{v_{i,j-1}^{k}-v_{i,j}^{k}}{h_2^2}\right)+\\
&\quad+\Big(I_{1,\sigma}(\overline{x}_{i,j}-\overline{h}_{\sigma,i,j}^{k})-I_{2,\sigma}(\overline{x}_{i,j})\Big)\;I_{1,y,\sigma}(\overline{x}_{i,j}-\overline{h}_{\sigma,i,j}^{k}). \qquad (9)
\end{aligned}$$

The notations are almost selfexplaining: for instance, τ is the time step size, h_1 and h_2 denote the pixel size in x and y direction, respectively, $u_{i,j}^k$ approximates u_σ in some grid point $\overline{x}_{i,j}$ at time $k\tau$, and $I_{1,x,\sigma}$ is an approximation to $G_\sigma * \frac{\partial I_1}{\partial x}$. $g_{i,j}^k = g\left(\|\nabla G_\sigma * I_1\|\right)\frac{\Phi'(\|\nabla u^k\|)}{\|\nabla u^k\|}(ih_1, ih_2)$ We calculate values of type $I_{1,\sigma}(\overline{x}_{i,j}-\overline{h}_{i,j}^{k})$ by linear interpolation,

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Figure 1: Numerical Experience. Up: we present two frames of a movie sequence. Down: We present the computed optical flow $u_{i,j}$ (left) and $v_{i,j}$ (right) using our method for the quadratic term. When the grey level of the images go to the white indicates a displacement of the object toward the right in the case of $u_{i,j}$ or toward the down in the case of $v_{i,j}$. In the same way, when the grey level of the images go to the black, it indicates a displacement of the object toward the left in the case of $u_{i,j}$ or toward the up in the case of $v_{i,j}$

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